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February 2016

Online at <https://mpra.ub.uni-muenchen.de/70663/>

MPRA Paper No. 70663, posted 16 May 2016 13:58 UTC

Uncertainty-Induced Dynamic Inefficiency and the Optimal Inflation Rate

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April 12, 2016

Abstract

I construct an overlapping-generations model of money with [Epstein and Zin \(1989\)](#) preferences and study how aggregate output uncertainty affects the optimal rate of inflation. When money only serves as savings instruments, I find that the optimality of *Friedman Rule* breaks up only if agents prefer late resolution of uncertainty. However, if an additional role of money as a medium of exchange is introduced, then the *Friedman Rule* becomes generally suboptimal regardless of agents' preferences for the timing of uncertainty resolution. The aggregate output uncertainty, nevertheless, crucially determines the level of optimal inflation rate in this case.

Keywords: money; overlapping generations; recursive preferences; optimal inflation

JEL Classification Numbers: E31, E52, E58

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“Policy decisions under uncertainty must take into account a range of possible scenarios about the state or structure of the economy, and those policy decisions may look quite different from those that would be optimal under certainty.” [Bernanke \(2007\)](#)

1 Introduction

Does *aggregate output uncertainty*, i.e., mean-preserving variance changes to aggregate output, matter for the level of optimal inflation rate? This question has probably never been more important than it is now for monetary authorities. Unfortunately, the existing literature has failed to provide a satisfactory answer to this question. For instance, the New-Keynesian literature has indeed studied the effect of “model uncertainty”, i.e., uncertainty about the parameters of the model, on the optimal inflation rate. However, because it heavily relies on zero lower bound risks, one and potentially incomplete conclusion has been drawn, i.e., the optimal inflation rate should increase in uncertainty to give more room for nominal interest rate policy, e.g., [Eggertsson and Woodford \(2003\)](#), [Billi \(2011\)](#), and [Coibion, Gorodnichenko, and Wieland \(2012\)](#).¹

Moreover, micro-founded monetary theories have remained silent on the aforementioned question. While they, from [Bewley’s \(1983\)](#) model of money to more recent search-based ones, e.g., see [Lagos, Rocheteau, and Wright \(2016\)](#) for an introduction, have focused on *idiosyncratic* uncertainty for the *essentiality* of money, *aggregate* uncertainty effects on the optimal inflation rate have been largely ignored. In other words, we still lack a monetary theory to analyze if *aggregate* output uncertainty has any effect on the optimal inflation rate, and if so, how such effects arise.²

The objective of this paper is therefore to propose a simple model of money that enables us to answer this question. In order to avoid the tractability problem associated with the aggregate uncertainty, I first adopt a two-period overlapping-generations (OLG) model of money. Second, I bring insights from a recent macro-finance literature that exploits an aggregate uncertainty based intertemporal marginal rate of substitution (IMRS) as in [Bansal and Yaron \(2004\)](#), i.e., [Epstein and Zin \(1989\)](#) (EZ) preferences. Two key advantages of EZ preferences are that agents’ aversion to cross-sectional and intertemporal risks become separated, and the certainty equivalent of future consumption value directly affects today’s

¹ See also [Bernanke \(2010\)](#) and [Mishkin \(2011\)](#) for skeptical views on this proposed relationship.

² Difficulties associated with these questions within heterogeneous models of money lie in tracking equilibria with aggregate shocks. The literature has barely started solving for equilibria with *idiosyncratic* risks analytically, e.g., see [Menzio et al. \(2013\)](#) and [Rocheteau, Weil, and Wong \(2015a\)](#). To incorporate aggregate shocks, one would seem to need different numerical mechanisms such as [Krusell and Smith \(1998\)](#). See [Chiu and Molico \(2014\)](#) who have recently developed a similar numerical technique for solving heterogeneous models of money under aggregate shocks.

utility. These features imply that agents' optimal intertemporal decision rules are *directly* affected by aggregate uncertainty, and can be derived in closed form.

The baseline model then considers a simple pure-currency endowment economy. The young receive fixed units of endowments, while the old receive an uncertain endowment. Fiat money becomes *essential* here because of its role as a sole savings instrument. Focusing only on a unique monetary stationary equilibrium, the model delivers interesting monetary policy implications. When agents' aversion to intertemporal risks are *relatively* greater, the *Friedman rule* becomes no longer optimal. The reason is that agents *oversave* in an attempt to minimize intertemporal inequality of consumption, i.e., uncertainty creates the so-called *dynamic inefficiency* in this OLG framework, e.g., [Diamond \(1965\)](#). Inflation up to a certain point can correct this inefficiency by creating another form of dynamic inefficiency, i.e., intergenerational transfers of endowments from the old to the young *via* lower rate of return on money savings.

On the contrary, when agents dislike cross-sectional risks more, the opposite result arises. Agents become more willing to transfer endowments towards today since they mind future consumption variation more than the intertemporal one. This implies that uncertainty and inflation generate dynamic inefficiency in the same direction so that the *Friedman rule* never achieves the social efficiency unless uncertainty vanishes, but guarantees the second best welfare outcome.

Despite these new insights, two obvious caveats exist in making a case against the *Friedman rule* using the baseline model. First, most of the EZ preferences based macro-finance literature is on the premise that agents' aversion to cross-sectional risks is larger (or agents prefer early resolution of uncertainty) in order to be consistent with empirical patterns for asset prices, e.g., see [Backus, Routledge, and Zin \(2004\)](#) and [Henriksen, Kydland, and Sustek \(2013\)](#). Second, the OLG model of money misses fundamental trading frictions that give rise to a medium of exchange (MOE) role of money. Thus, any policy recommendation from the OLG model is at best incomplete.

To address these concerns, I extend the baseline model to include decentralized trading mechanism in which money endogenously emerges as an MOE. To formalize this idea, the extended model adopts the [Lucas's \(1972\)](#) island framework augmented with decentralized markets. In the generalized model, an additional consumption good, referred to as *special good*, is introduced to generate the additional MOE role for money. In consequence, young agents now face a problem of portfolio choice between money savings and money holdings for spending on the special good.

The extended model then delivers a richer set of policy implications, and opens the possibility of breaking the *Friedman rule* regardless of agents' preferences for the timing of

uncertainty resolution. When agents' aversion to intertemporal risks is relatively higher, agents over-accumulate money savings for the same reason in the baseline model. This, in turn, endogenously affects the agent's optimal portfolio to the extent that cash balances for the purpose of buying the special good get inefficiently reduced. Accordingly, welfare effects of inflation become non-trivial as opposed to the baseline model. A higher inflation lowers the rate of return on money savings, and therefore can mitigate the intertemporal misallocations caused by the over-money-savings. We call this a *positive redistributive effect* of inflation. On the contrary, a higher inflation simultaneously worsens the inefficiency caused by under-cash-spending for the special good. We call this as a *negative price effect* of inflation. Eventually, the optimal inflation rate depends on which effect relatively dominates.

Numerical examples show that a positive inflation up to a certain threshold can indeed make the positive redistributive effect dominant if such distortions, i.e., over-money-savings and under-cash-spending, were initially severe enough. Otherwise, the price distortion effect always dominates. Consequently, the *Friedman rule* achieves the second best. These results are intuitive because the marginal (positive) redistributive effect of inflation is bigger when the intertemporal allocation is already severely distorted. This mechanism draws an interesting implication on the link between the aggregate output uncertainty and the optimal inflation rate. Since a higher uncertainty worsens intertemporal misallocations in the first place, the positive redistributive effect of inflation gets stronger. Thus, the optimal inflation rate ought to increase in the aggregate output uncertainty in this case.

By the same token, the trade-off between the price and redistributive effect of inflation works the opposite way when agents' aversion to cross-sectional risks is higher. The reason is as follows. The aggregate uncertainty induces agents to under-accumulate money savings for the same reason in the baseline model. This implies that a higher inflation worsens intertemporal misallocations, causing a *negative redistributive effect* of inflation this time. At the same time, the portfolio effect ensures that over-cash-spending prevails in equilibrium, thereby making the price effect of a higher inflation *positive* this time. In consequence, two interesting implications can be drawn here. First, unlike the baseline model, the *Friedman rule* can be suboptimal even under the agent's preferences for the early resolution of uncertainty when the *positive price effect* of inflation is strong enough. The second, and more interesting result, is that a higher aggregate output uncertainty can lead to a *lower* optimal inflation rate in this case. This is due to the fact that the marginal *negative redistributive effect* of inflation gets stronger as higher uncertainty distorts intertemporal misallocations more.

I do not intend to thoroughly review a vast literature of money studying optimal monetary policy. Interested readers may refer to many excellent review papers such as [Kocherlakota](#)

(2005) and Antinolfi, Azariadis, and Bullard (2016). I only provide a brief review of studies that are related to the current paper in terms of the methodology and contribution.

To begin with, there are indeed existing OLG models of money making a case against the *Friedman rule*, e.g., Schreft and Smith (1997); Schreft and Smith (2002); Smith (2002); Bhattacharya, Haslag, and Martin (2005). Basically, they introduce financial intermediation and limited communication to bring about heterogeneity and the suboptimality of the *Friedman rule*.³ Unlike them, the suboptimality in this paper emerges through an interaction between aggregate output uncertainty and optimal intertemporal decisions by agents.

This paper also pertains to search-based models of money that emphasize redistributive effects of inflation. Papers with analytically tractable models include but not limited to Berensten, Camera, and Waller (2005); Menzio, Shi, and Sun (2013); Rocheteau et al. (2015b); Rocheteau et al. (2015a), while the ones based on numerical methods are, for example, Molico (1997); Kim and Lee (2008); Chiu and Molico (2010); Chiu and Molico (2014). These papers generate stationary equilibria with non-degenerate distribution of money holdings, thereby inducing inflation to enhance risk sharing, i.e., a case against the *Friedman rule*.⁴ However, as already argued, they have yet to provide a framework for the aggregate uncertainty based monetary policy analysis. In this aspect, the current model contributes by proposing a very simple and tractable benchmark. Further, the current model is novel because a portfolio decision of money holdings for a store of value and a medium of exchange is considered.

There are different but similarly related empirical studies on the redistribution effects of inflation. Doepke and Schneider (2006) show that inflation substantially affects nominal wealth distribution through the role of money as a unit of account. On the contrary, the current model emphasizes on other roles of money, i.e., as a store value and a medium of exchange, for the wealth redistribution. Erosa and Ventura (2002) argue that inflation hurts poor households more since they tend to over-hold cash relative to other forms of financial assets. Unlike them, asymmetric inflation tax is levied upon different money holdings, i.e., either for savings and spending, rather than different households.

The remainder of the paper proceeds as follows. Section 2 describes the physical environment in the baseline model. Section 3 studies efficient and competitive equilibrium allocations in the baseline model. Section 4 introduces additional environments and analyses the constrained efficient and competitive equilibrium allocations. Section 5 concludes.

³ Haslag and Martin (2003) show that this suboptimality hinges upon the existence of some instruments that allow intergenerational transfers.

⁴ Recent studies, e.g., Lagos and Zhang (2015), Geromichalos and Herrenbrueck (2016), and Geromichalos and Jung (2016), explore such redistributive effects within secondary over-the-counter asset markets using money as MOE.

2 The baseline model

We consider a simple version of OLG model advocated by [Wallace \(1980\)](#). Time is infinite and discrete. In each period t a unit measure of agents are born who live only two periods, i.e., t and $t + 1$. An agent born in period t has preferences of [Epstein and Zin \(1989\)](#) (EZ) type, $U(c_t, c_{t+1})$, given by the following form.

$$U(c_t, c_{t+1}) = [c_t^{1-\rho} + [R_t(c_{t+1})]^{1-\rho}]^{\frac{1}{1-\rho}}$$

$$\text{where } R_t(c_{t+1}) = (E_t [c_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}, \rho > 0.$$

Intuitively, EZ preferences are defined as a CES (Constant Elasticity of Substitution) aggregate of current (known) consumption and a certainty equivalent $R_t(c_{t+1})$ of tomorrow's consumption. Consequently, ρ and γ are interpreted as the inverse of the intertemporal elasticity of substitution and the risk aversion parameter respectively. A key feature of the EZ preferences is that the two parameters, ρ and γ , can be separated while ρ is always pinned by γ , i.e., $\rho = \gamma$, in a standard time-separable CRRA (Constant Relative Risk Aversion) utility case. In other words, this distinctive feature of the EZ preferences allows agents to differ in terms of an aversion to cross-sectional and intertemporal risks. When agents have a *relatively* higher degree of aversion to cross-sectional (intertemporal) risks, i.e., $\gamma > \rho$ ($\gamma < \rho$), they are often regarded as ones who have preferences for early (late) resolution of uncertainty in the literature.

The rest of environments follows a standard OLG model of money closely. Agents in their first period of life will be called *young* while we use the term *old* for those who are in their second period of life. We also assume that a unit measure of initial old agents, who live only for one period, exists in the very first period 1. We call these agents as *initial old*. Further, they are assumed to be collectively endowed with M_0 units of perfectly divisible, intrinsically useless, and government issued *fiat* money. Importantly, an endowment economy is assumed. All young agents receive fixed units of the perishable consumption good, x . However, old agents have idiosyncratic endowment risks. That is when agents become *old*, they receive i.i.d. endowment shocks, ε units of the consumption good. For simplicity, we assume that ε follows a uniform distribution, $\mathcal{U}(y - b, y + b)$ where $y \geq b$ and $x \geq (y + b)$.

Time discounting and population growth are ruled out for simplicity. The government's policy is implemented through lump-sum money transfers to old agents in each period. We denote τ_t as the lump-sum transfer that each old agents receives in period t , in terms of the period t consumption good. We also let M_t denote the total money supply in period t . No other forms of tax or transfers are assumed to be feasible. That is the lump-sum money

transfers are the only available instrument for the government policy in this economy. Given this restriction, the government budget constraint implies $\varphi_t(M_t - M_{t-1}) = \tau_t$, $\forall t$ where φ_t denotes the real price of money in period t . Finally, the government is assumed to increase or decrease money supply at a constant rate each period, i.e., $M_t = \mu M_{t-1}$, $\forall t$ with $\mu \geq 1$.⁵ In sum, the government budget constraint and the monetary policy rule implies the following condition every period.

$$\varphi_t M_t (1 - 1/\mu) = \tau_t \quad \forall t. \quad (1)$$

3 Efficiency and competitive equilibrium

We first study a social planner problem in this economy. In doing so, we only focus on stationary allocations, i.e., the planner is constrained in such a way that he/she can only choose stationary allocations in this economy. Let (c_y^*, c_o^*) denote the stationary allocations that the social planner chooses to maximize the welfare of agents born in generations $\forall t$. Then, it must be a solution to the following problem.

$$\begin{aligned} \max_{c_y^*, c_o^*} & \left[(c_y^*)^{1-\rho} + ([c_o^*]^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}} \\ \text{s.t.} & \quad c_y^* + c_o^* = x + y, \end{aligned} \quad (2)$$

where the aggregate (resource) constraint indicates that total consumption of young agents plus total consumption of old agents can not exceed the total endowment in each period. Note that the expectation operator inside the EZ preferences is eliminated due to no uncertainty on c_o^* . The following lemma summarizes the socially optimal stationary allocations of consumptions by young and old agents.

Lemma 1 *The socially efficient stationary allocations satisfy the following condition.*

$$c_y^* = c_o^* = \frac{x + y}{2}.$$

Proof. The proof follows easily from the first order condition to the problem (2), and it is, therefore, omitted. ■

The efficient stationary allocations are characterized by equal division of endowments between young and old agents. This follows directly from that fact that the objective function

⁵ The *Friedman* rule corresponds, therefore, to $\mu = 1$, i.e., zero inflation, in this economy.

(indifference curve) is convex to the origin and the absolute slope of resource constraint equals to one.

Next, we study the properties of a competitive equilibrium in this economy. First, an agent born in period t faces the following problem.

$$\begin{aligned} \max_{c_t, c_{t+1}} & \left[(c_t)^{1-\rho} + E_t \left[(c_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}} \\ \text{s.t.} & \quad c_t + \varphi_t m_t = x, \\ \text{and} & \quad c_{t+1} = \varphi_{t+1} m_t + \tau_{t+1} + \varepsilon_{t+1}. \end{aligned} \quad (3)$$

where m_t denotes the nominal quantity of money held by an agent born in period t , and ε_{t+1} denotes the stochastic endowment that the old receive. Intuitively, agents face uncertainty with regard to endowments when they become old. Thus, fiat money serves as a savings instrument for young agents. This admits intuitive explanation for the two constraints in problem (3). The first one refers to a budget constraint for young agents in their first period of life, and the second one simply says that consumption when old must be financed by money savings from previous period, lump-sum money transfers by the government, and idiosyncratic endowment shocks.

To simplify the above problem, substitute for the two constraints in the objective function. This leads one to obtain

$$\max_{m_t} U(x - \varphi_t m_t, \varphi_{t+1} m_t + \tau_{t+1} + \varepsilon_{t+1}).$$

Assuming an interior solution, the first-order condition is given by

$$\frac{\varphi_t}{\varphi_{t+1}} = \frac{U_2(x - \varphi_t m_t, \varphi_{t+1} m_t + \tau_{t+1} + \varepsilon_{t+1})}{U_1(x - \varphi_t m_t, \varphi_{t+1} m_t + \tau_{t+1} + \varepsilon_{t+1})} \equiv Q_{t,t+1},$$

where $U_j(c_t, c_{t+1}), \forall j \in \{1, 2\}$ denotes the first partial derivative of the EZ utility function with respect to the j th argument, and $Q_{t,t+1}$ denotes the intertemporal marginal rate of substitution (IMRS). Using the characteristics of the EZ preferences the following lemma summarizes individual optimal choice by young agents.

Lemma 2 *Let $\varphi_{t+1} m_t + \tau_{t+1} \equiv h(m_t)$. Given aggregate real money prices $\{\varphi_t, \varphi_{t+1}\}$ and individual endowment shocks ε_{t+1} , the young agents' individual optimal choice of m_t must*

satisfy the following condition. $\varphi_t/\varphi_{t+1} = Q_{t,t+1}(m_t, \varepsilon_{t+1}) \forall t$ where

$$Q_{t,t+1}(m_t, \varepsilon_{t+1}) = \left[\frac{h(m_t) + \varepsilon_{t+1}}{x - \varphi_t m_t} \right]^{-\rho} \left[\frac{h(m_t) + \varepsilon_{t+1}}{E_t[(h(m_t) + \varepsilon_{t+1})^{1-\gamma}]^{\frac{1}{1-\gamma}}} \right]^{\rho-\gamma},$$

and $E_t[(h(m_t) + \varepsilon_{t+1})^{1-\gamma}] = \frac{(h(m_t) + y + b)^{(2-\gamma)} - (h(m_t) + y - b)^{(2-\gamma)}}{2b(2-\gamma)}.$

Proof. See the appendix. ■

A key point to note here is that uncertainty regarding future endowments directly affects IMRS under $\rho \neq \gamma$ as can be verified in Lemma 2. Intuitively, the EZ preferences depend on the certainty equivalent of tomorrow's consumption value instead of the expected value of tomorrow's consumption. This implies that whenever the variance of tomorrow's consumption changes, the certainty equivalent of tomorrow's consumption value changes accordingly which in turn affects the relative value of future consumption, i.e., IMRS. This particular mechanism has far reaching implications on the welfare of this economy in equilibrium, to be discussed in much detail later. Lastly, note that this mechanism disappears as long as $\rho = \gamma$, i.e., under the standard time-separable CRRA utility case. In other words, uncertainty with regard to aggregate output does not affect consumption allocations under a standard OLG model of money with a conventional utility functional form.

Now, we describe competitive equilibrium, with a special focus on the effects of monetary policy on welfare. First, we restrict attention to symmetric, monetary, and stationary equilibrium where $\varphi_t > 0, \forall t$, all young agents in each period choose the same real money balances, and the real variables of the model remain constant over time. Then, the definition of symmetric, monetary, and stationary equilibrium is given by

Definition 1 *A competitive, symmetric, monetary, and stationary equilibrium is a list $\{Z, c_y, c_o\}$, where $Z = Z_t = \varphi_t M_t, \forall t$ and $\{c_y, c_o\} = \{x - Z, Z + y\}$. The equilibrium real money balance Z satisfies Lemma 2 given that $\varphi_t/\varphi_{t+1} = \mu$, $\varepsilon_{t+1} = E[\varepsilon] = y$, and $\varphi_t m_t = \varphi_{t+1} m_{t+1} + \tau_{t+1} = Z$.*

This definition gives rise to the following log-linearized equation that Z must satisfies, i.e., $G(Z, \mu, x, y, b) = 0$, where

$$G(Z, \mu, x, y, b) \equiv \ln(\mu) + \rho \{ \ln(Z + y) - \ln(x - Z) \} - (\rho - \gamma) \ln(Z + y) - \frac{\gamma - \rho}{1 - \gamma} \left\{ \ln \left[\frac{(Z + y + b)^{2-\gamma}}{(2 - \gamma)} - \frac{(Z + y - b)^{2-\gamma}}{(2 - \gamma)} \right] - \ln(2b) \right\}. \quad (4)$$

Equation (4) allows us to conduct various comparative static analyses on stationary welfare of the economy. In order to emphasize the effects of uncertainty regarding endowment shocks, i.e., the variance of ε , we categorize the equilibrium into three different cases: 1. $\rho = \gamma$, i.e., indifferent preferences for the resolution of uncertainty, 2. $\rho > \gamma$, i.e., preferences for late resolution of uncertainty, and 3. $\rho < \gamma$, i.e., preferences for early resolution of uncertainty.

The first case where $\rho = \gamma$, i.e., the utility functional form is time-separable CRRA, is straightforward. As in a standard OLG model of money, uncertainty, i.e., changes in b , has no effects on the stationary allocation, and the *Friedman Rule*, achieves the social efficiency. These can be easily understood by setting $\rho = \gamma$ in eq.(4). Intuition is that $\rho = \gamma$ leads to complete equalization of expected future consumption and its certainty equivalent value, which in turn implies no uncertainty effects on the IMRS. In addition, as soon as μ exceeds one, i.e., positive inflation prevails in stationary equilibrium, the rate of return on money savings decreases. Therefore, young agents save less and inefficiently consume too much compared to the social optimum. In other words, money creation causes *dynamic inefficiency* to the extent that intergenerational transfers of resources from the old to the young harm the social welfare.

When agents' level of aversion to cross-sectional and intertemporal risks differ, things get a lot more interesting. First, the following proposition summarizes comparative static analyses under the case where agents prefer late resolution of uncertainty, i.e., $\rho > \gamma$.

Proposition 1 *Consider the second case where $\rho > \gamma$. Let Z_{EZ} denote real money balances in stationary equilibrium. Likewise, let Z_{SP} denote the social planner's (implied) resource transfers from the young to the old, i.e., $Z_{SP} \equiv (x - y)/2$. A unique stationary monetary equilibrium exists, i.e., $\exists! Z_{EZ}$, only if $\ln \mu < \bar{\mu}$, where*

$$\bar{\mu} = \frac{\gamma - \rho}{1 - \gamma} \left\{ \ln \left[\frac{(y + b)^{2-\gamma} - (y - b)^{2-\gamma}}{2b(2 - \gamma)} \right] \right\} - \gamma \ln y + \rho \ln x.$$

The followings then hold true in the unique stationary monetary equilibrium: $Z_{EZ} > Z_{SP}$, $\partial Z_{EZ}/\partial \mu < 0$, and $\partial Z_{EZ}/\partial b > 0$. Lastly, there exists a unique money growth rate μ^ greater than one that achieves the social optimum, and μ^* has the following closed form solution.*

$$\mu^* = \left[\frac{x - y}{2} \right]^{\rho - \gamma} \left[\frac{\left(\frac{x+b}{2} \right)^{2-\gamma} - \left(\frac{x-b}{2} \right)^{2-\gamma}}{2b(2 - \gamma)} \right]^{\frac{\gamma - \rho}{1 - \gamma}} > 1,$$

where $\partial \mu^/\partial b > 0$, $\partial \mu^*/\partial y < 0$, and $\partial \mu^*/\partial (x - y) > 0$.*

Proof. See the appendix. ■

Proposition 1 can be intuitively understood through the lens of agents' uncertainty aversion. Under the EZ preferences with $\rho > \gamma$ agents' aversion to intertemporal risks is greater than that to cross-sectional risks. Loosely speaking, agents dislike intertemporal inequality more than cross-sectional inequality among old agents. This in turn means that agents are willing to transfer resources from the young to the old a lot more aggressively compared to the social optimum, i.e., $Z_{EZ} > Z_{SP}$.

By the same token, the fact that higher uncertainty regarding future endowment shocks leads agents to save more, i.e., $\partial Z_{EZ}/\partial b > 0$, can be easily explained. Under the EZ preferences, agents' utility comes from current consumption and the certainty equivalent of future consumption. Since higher uncertainty, i.e., a higher b , reduces the certainty equivalent value, other things being equal, the higher uncertainty would *effectively* lead to more intertemporal inequality. Thus, under preferences for late resolution of uncertainty, i.e., $\rho > \gamma$, agents would transfer more consumption to the old in response to a higher level of uncertainty regarding future endowment shocks.

However, the effect of inflation on money savings would be completely opposite. The idea is straightforward as in the first case. A higher money growth rate depresses the rate of return on money savings, and therefore lowers equilibrium real money balances, i.e., $\partial Z_{EZ}/\partial \mu < 0$. In sum, uncertainty, i.e., $b > 0$, and inflation, i.e., $\mu > 1$, both generate dynamic inefficiency but in opposite direction. The former tends to generate over-money-savings while the latter does generate under-money-savings.

This at the end of the day rationalizes why optimal inflation could be a positive level in this framework. Dynamic inefficiency caused by uncertainty can be corrected by money creation which also generates dynamic inefficiency but exactly in opposite direction. This intuition can be applied to understand $\partial \mu^*/\partial b > 0$. Higher uncertainty leads to over-money-savings to a greater extent. In order to offset this force, an opposite force generating more under-money-savings, i.e., higher inflation, is needed. Furthermore, the optimal inflation rate falls as the mean of future endowment shocks increases, i.e., $\partial \mu^*/\partial y < 0$. This result is again intuitive. A higher y means less intertemporal inequality. Thus, under $\rho > \gamma$ agents' desire to save also gets weaker. Since Z_{EZ} gets smaller and closer to Z_{SP} , optimal inflation rate should fall as well.

Next, we move on to the third case, $\rho < \gamma$. When agents prefer early resolution of uncertainty, the economy behaves in somewhat opposite way to the second case. Proposition 2 summarizes comparative static analyses under the case where agents prefer early resolution of uncertainty, i.e., $\rho < \gamma$.

Proposition 2 *Consider the third case where $\rho < \gamma$. A unique stationary monetary equilibrium exists, i.e., $\exists! Z_{EZ}$ only if $\ln \mu < \bar{\mu}$. When $\ln \mu \geq \bar{\mu}$, stationary equilibrium is*

either non-monetary or monetary, but exhibits multiplicity. Focusing on the unique stationary monetary equilibrium, the followings then hold true: $Z_{EZ} < Z_{SP}$, $\partial Z_{EZ}/\partial \mu < 0$, and $\partial Z_{EZ}/\partial b < 0$. Lastly, the Friedman rule, i.e., $\mu = 1$, does not achieve the social optimum, but does guarantee the second-best.

Proof. See the appendix. ■

In order to understand the Proposition 2 one needs to recall intuition from Proposition 1. First, $\rho < \gamma$ effectively means that agents value uncertain future consumption a lot less than certain current consumption, i.e., they dislike cross-sectional variation in consumption among the old more than intertemporal variation in consumption. Thus, agents are less willing to transfer resources from the young to the old compared to the social optimum, i.e., $Z_{EZ} < Z_{SP}$.

The effect of uncertainty on real money balances is also opposite to the second case. Under $\rho < \gamma$, higher uncertainty, i.e., a higher b , makes agents value future consumption less than the current one because they *relatively* dislike cross-sectional variation in future consumption more. Thus, agents would reduce real money balances, i.e., less savings, in response to a higher uncertainty, i.e., $\partial Z_{EZ}/\partial b < 0$.

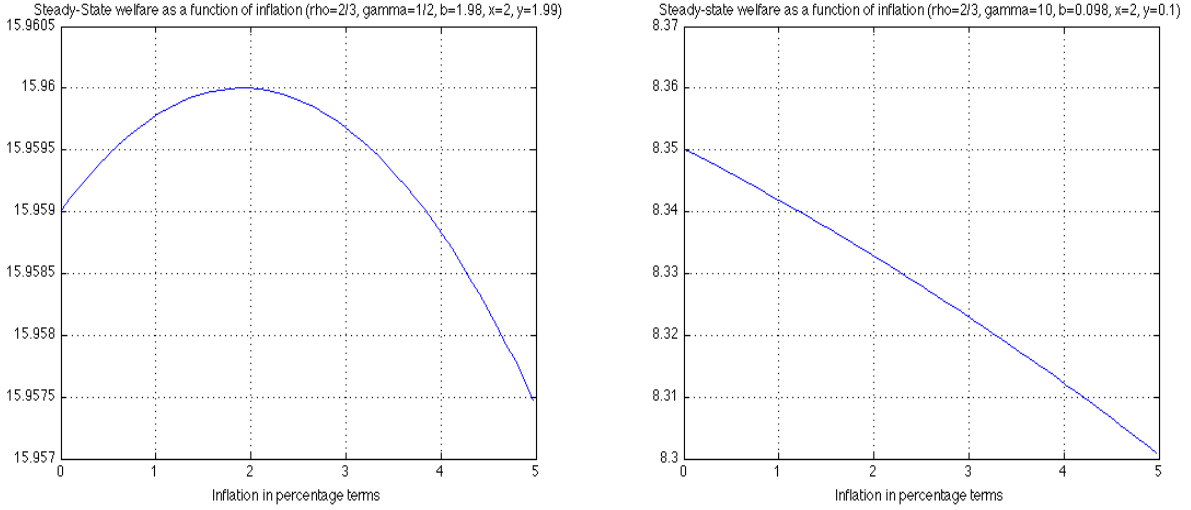
A higher inflation would still reduce real money balances as in the second case, i.e., $\partial Z_{EZ}/\partial \mu < 0$. This is because changes in the rate of return on money savings does not affect agents' aversion to either cross-sectional or intertemporal risks. To sum up, a higher inflation and a higher level of uncertainty both leads to under-money-savings when agents prefer early resolution of uncertainty. In other words, money creation and uncertainty generates dynamic inefficiency in the same direction. This inevitably implies that even *Friedman rule* does not achieve the social efficiency under uncertainty, but guarantees the second best welfare outcome.

Figure 1 illustrates numerical examples on how the welfare of this economy responds to changes in inflation rate at stationary equilibrium. The steady-state welfare is defined as the sum of all agents' net utilities in the unique stationary monetary equilibrium. The left panel corresponds to the second case, i.e., $\rho > \gamma$, while the right panel shows the result in the third case, i.e., $\rho < \gamma$.

Lessons from this simple model is clear. Welfare effects of inflation critically hinges upon how agents perceive uncertainty risks. Monetary authority can achieve the first-best through money creation only if people prefer late resolution of uncertainty. Otherwise, the first-best is not feasible, but to achieve the second-best they should pursue a *Laissez-Faire* policy, i.e., zero money growth. A final point is that most macro-finance literature heavily based upon the EZ preferences assumes $\rho < \gamma$ because the latter generates asset pricing implications

consistent with empirical observations (see [Bansal and Yaron \(2004\)](#) and related literature). However, this does not necessarily have to mean that the *Friedman rule* is always the best policy to pursue even after taking account of uncertainty effects, which is a topic of the next section.

Figure 1: Numerical examples for welfare analysis in the baseline model



4 The extended model

Critiques against the OLG model of money often point that it is not explicit about monetary exchange, potentially the most important source of the *essentiality* of money. To put it differently, money does not serve a medium of exchange (MOE) role in the OLG model. To remedy this deficiency search models with microfoundations of monetary exchange have emerged and recently become the workhorse paradigm for monetary theory (see [Lagos et al. \(2016\)](#) for a survey of recent search based monetary theories). For this reason we modify model environments to incorporate additional role of money as MOE, and study if this modification alters any welfare implications of inflation.

To begin with, we adopt a modified version of the [Lucas's \(1972\)](#) island model. The economy consists of one main island at the center and a unit measure of periphery islands. An agent called *seller* is born in each periphery island each period, and live for only one period. We assume that each seller is endowed with a technology to produce *special goods* with linear disutility of labor. These special goods are perishable with a one-period life and are not portable across islands. In addition, sellers can move around islands without any physical frictions, and get the linear utility from consuming *general goods*, endowed only to

agents called *household* who lives at the main island.

Household in this economy, to some extent, resembles the agent in the previous model. Each period, a unit measure of household is born in the main island, and lives only for two periods. Each young household is endowed with fixed units of perishable and not-portable-across-islands general good, x . Yet, old households receive random units of general good, ε following a uniform distribution, $\mathcal{U}(y - b, y + b)$ where $y \geq b$ and $x \geq (y + b)$. Unlike the previous economy, this household consists of two individuals called *worker* and *shopper*. Workers are not allowed to leave the main island, and get the utility only from general goods consumed. Shoppers are free to move around islands, and young shoppers never get the consumption utility. Yet, old shoppers get the utility only from consuming special goods, which they need to acquire from sellers living in periphery islands. Lastly, the rest of environments are identical to the previous model. Monetary policy is implemented in the same manner as the previous model, i.e., lump-sum money transfers to old households in the main island with a constant growth rate, $\mu \geq 1$. Time discounting and population growth are ruled out.

Altogether, these features of the economy creates the motivation for bilateral trading between shoppers and sellers every period. Following standard search based monetary theories, we assume limited commitment and anonymity within bilateral meetings. This rules out any kind of credit arrangement between shoppers and sellers. That is in each period only old shoppers leave the main island and search for sellers living in periphery islands. For simplicity we assume a perfect matching, i.e., every periphery island receives only one old shopper each period. Barter is not feasible in this framework since general goods are not portable across islands. This consequently gives rise to money serving an additional MOE role in this economy. To avoid complexity we adopt *take-it-or-leave-it* offer by (old) shoppers to sellers as pricing protocol within a pair-wise trade. Assuming old shoppers can never come back to the main island once they leave, bargaining solutions are trivial. Old shopper always gives up all of her real balances brought up to the meeting, and young seller produces exactly the same amount of special goods as the real money balances he or she receives.

Lastly, a household born in period t has preferences of [Epstein and Zin \(1989\)](#) (EZ) type, $U(c_t, s_{t+1}, c_{t+1})$, given by the following form.

$$U(c_t, s_{t+1}, c_{t+1}) = [c_t^{1-\rho} + s_{t+1}^{1-\rho} + [R_t(c_{t+1})]^{1-\rho}]^{\frac{1}{1-\rho}}$$

$$\text{where } R_t(c_{t+1}) = (E_t [c_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}, \rho > 0.$$

c_t and c_{t+1} denote the amount of general goods consumed by the worker in period t and

$t + 1$, while s_{t+1} denotes the amount of special goods consumed by the shopper in period $t + 1$. Figure 2 illustrates the timing of key events.

Figure 2: Timing of Events

Period t	Period $t + 1$
<ul style="list-style-type: none"> • Young households are born in the main island, and choose a portfolio of money savings and cash holdings for purchasing special goods. • A seller is born in each periphery island, and trade with an old shopper using cash. • After the bargaining is done, the seller moves to the main island and consumes general goods and then die. 	<ul style="list-style-type: none"> • Old workers consume general goods using money savings, and then die. • Old shoppers visit sellers and trade using cash as MOE. They consume special goods and die. • Sellers repeat the same action as in period t.

4.1 Efficiency and competitive equilibrium

As in the previous model, we only focus on stationary allocations. Let (c_y^*, s^*, c_o^*, n^*) denote the stationary allocations (general goods consumed by young workers, special goods consumed by old shoppers, general goods consumed by old workers, and general goods consumed by sellers respectively) that the social planner chooses to maximize the welfare of agents born in generations $\forall t$. Yet, unlike the previous model, we call the social planner's solution as the constrained efficient stationary allocation because we assume that the planner here is subject to restrictions of the physical environment, such as the trading protocol in each periphery island. Then, the planner's solution solves for the following problem.

$$\begin{aligned}
& \max_{c_y^*, c_o^*} \left\{ \left[(c_y^*)^{1-\rho} + (s^*)^{1-\rho} + ([c_o^*]^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}} + [n^* - s^*] \right\} \\
& \text{s.t. } c_y^* + c_o^* + n^* = x + y \\
& \quad \text{and } s^* = n^*,
\end{aligned} \tag{5}$$

where the first aggregate (resource) constraint implies that total general goods consumed by households and sellers must be same as total endowments of the general good in each period. The second aggregate (resource) constraint simply tells that total special goods consumed by (old) shoppers equal to total special goods produced by sellers each period. Note that the latter equals to the former due to the *take-it-or-leave-it* offer, which also explains the second linear part in the objective function. The following lemma summarizes the socially optimal stationary allocations of consumptions by households and sellers.

Lemma 3 *The constrained efficient stationary allocations satisfy the following condition.*

$$c_y^* = s^* = c_o^* = n^* = \frac{x + y}{3}.$$

Proof. The proof follows easily from the first order condition to the problem (5), and it is, therefore, omitted. ■

Intuition behind this constrained efficient stationary allocation follows from the previous model in a similar way.

We proceed to competitive equilibrium analysis. Unlike the previous model, a young household needs to acquire money balances not only for smooth general good consumption but also for purchasing special goods when old. This gives rise to a portfolio choice problem for the young household, i.e., m_t^c (money savings for general good consumption) and m_t^s (cash holdings for purchasing special goods).⁶ Then, the problem faced by a household born in period t is given by

$$\max_{m_t^c, m_t^s} U(x - \varphi_t m_t^c - \varphi_t m_t^s, \varphi_{t+1} m_t^s + \tau_{t+1}^s, \varphi_{t+1} m_t^c + \tau_{t+1}^c + \varepsilon_{t+1}),$$

where the first, second, and third input refers to c_t , s_{t+1} , and c_{t+1} respectively. Note that $\tau_{t+1}^k = (m_t^k/m_t)\tau_{t+1}$, $k \in \{c, s\}$ where m_t denotes total money balances held by the young household. τ_{t+1} denotes the lump-sum transfer that the young household is expected to receive in period $t + 1$, and satisfies eq.(1).

Assuming an interior solution, the FOC comprises of a system of two equations given by

$$\frac{\varphi_t}{\varphi_{t+1}} = \frac{U_3(x - \varphi_t m_t^c - \varphi_t m_t^s, \varphi_{t+1} m_t^s + \tau_{t+1}^s, \varphi_{t+1} m_t^c + \tau_{t+1}^c + \varepsilon_{t+1})}{U_1(x - \varphi_t m_t^c - \varphi_t m_t^s, \varphi_{t+1} m_t^s + \tau_{t+1}^s, \varphi_{t+1} m_t^c + \tau_{t+1}^c + \varepsilon_{t+1})} \equiv Q_{t,t+1}, \quad (6)$$

$$1 = \frac{U_3(x - \varphi_t m_t^c - \varphi_t m_t^s, \varphi_{t+1} m_t^s + \tau_{t+1}^s, \varphi_{t+1} m_t^c + \tau_{t+1}^c + \varepsilon_{t+1})}{U_2(x - \varphi_t m_t^c - \varphi_t m_t^s, \varphi_{t+1} m_t^s + \tau_{t+1}^s, \varphi_{t+1} m_t^c + \tau_{t+1}^c + \varepsilon_{t+1})}, \quad (7)$$

where $U_j(c_t, s_{t+1}, c_{t+1})$, $\forall j \in \{1, 2, 3\}$ denotes the first partial derivative of the EZ utility function with respect to the j th argument, and $Q_{t,t+1}$ in eq.(6) denotes the intertemporal marginal rate of substitution (IMRS) for general goods. Equation (7) implies the intra-temporal optimality between s_{t+1} and c_{t+1} . Using the characteristics of the EZ preferences the following lemma summarizes individual optimal choice by the young household.

Lemma 4 *Let $\varphi_{t+1} m_t^c + \tau_{t+1}^c \equiv h(m_t^c)$ and $\varphi_{t+1} m_t^s + \tau_{t+1}^s \equiv h(m_t^s)$. Given aggregate real money prices $\{\varphi_t, \varphi_{t+1}\}$ and individual endowment shocks ε_{t+1} , the young household's optimal*

⁶ We assume the endowment shocks are realized after shoppers leave the main island in order for a household to choose a portfolio *ex-ante*.

portfolio choice of $\{m_t^c, m_t^s\}$ must satisfy the following conditions

$$\begin{aligned}
1. & \frac{\varphi_t}{\varphi_{t+1}} = Q_{t,t+1}(m_t^c, m_t^s, \varepsilon_{t+1}) \quad \forall t, \\
2. & h(m_t^s) = [x - \varphi_t m_t^c - \varphi_t m_t^s]^{\gamma/\rho} \left[E_t [(h(m_t^c) + \varepsilon_{t+1})^{1-\gamma}]^{\frac{1}{1-\gamma}} \right]^{(\rho-\gamma)/\rho} \quad \forall t, \\
\text{where } & Q_{t,t+1}(m_t^c, m_t^s, \varepsilon_{t+1}) = \left[\frac{h(m_t^c) + \varepsilon_{t+1}}{x - \varphi_t m_t^c - \varphi_t m_t^s} \right]^{-\rho} \left[\frac{h(m_t^c) + \varepsilon_{t+1}}{E_t [(h(m_t^c) + \varepsilon_{t+1})^{1-\gamma}]^{\frac{1}{1-\gamma}}} \right]^{\rho-\gamma}, \\
\text{and } & E_t [(h(m_t^c) + \varepsilon_{t+1})^{1-\gamma}] = \frac{(h(m_t^c) + y + b)^{(2-\gamma)} - (h(m_t^c) + y - b)^{(2-\gamma)}}{2b(2-\gamma)}.
\end{aligned}$$

Proof. Proof for the intertemporal optimality follows easily from Proof for Lemma 2. The intra-temporal optimality can be easily derived from the fact that $U_2 = U^\rho s_{t+1}^{-\rho}$. ■

Interpretation of Lemma 4 follows similarly from Lemma 2 except for the second intra-temporal optimality. Now we directly proceed to competitive equilibrium. As before, we restrict attention to the symmetric monetary, and stationary equilibrium, which is defined as follows.

Definition 2 A competitive, symmetric, monetary, and stationary equilibrium is a list $\{Z^c, Z^s, n, s, c_y, c_o\}$, where $Z_t^c \equiv \varphi_t m_t^c = h(m_t^c) = Z^c \forall t$, $Z_t^s \equiv \varphi_t m_t^s = h(m_t^s) = Z^s \forall t$, $Z^s + Z^c = \varphi_t M_t \forall t$, and $\{c_y, c_o, s, n\} = \{x - Z^s - Z^c, Z^c + y, Z^s, Z^s\}$. The equilibrium real money balances $\{Z^c, Z^s\}$ satisfy Lemma 4 given that $\varphi_t/\varphi_{t+1} = \mu$ and $\varepsilon_{t+1} = E[\varepsilon] = y$.

Following this definition, a system of two log-linearized equations that $\{Z^c, Z^s\}$ must satisfy emerges.

$$\ln(\mu) - \rho \ln(x - Z^c - Z^s) + \gamma \ln(Z^c + y) = (\gamma - \rho) \ln(R(Z^c + y)) \quad (8)$$

$$\ln(Z^s) = (\gamma/\rho) \ln(Z^c + y) + \{(\rho - \gamma)/\rho\} \ln(R(Z^c + y)), \quad (9)$$

$$\text{where } \ln(R(Z^c + y)) = \frac{1}{1-\gamma} \left\{ \ln \left[\frac{(Z^c + y + b)^{2-\gamma}}{2b(2-\gamma)} - \frac{(Z^c + y - b)^{2-\gamma}}{2b(2-\gamma)} \right] \right\}.$$

As in the previous model, we conduct comparative static analyses based on the three cases regarding agents' preferences for uncertainty resolution. The first case where $\rho = \gamma$ admits the same welfare implication of inflation as before. When agents are indifferent to the timing of uncertainty resolution, the *Friedman rule* achieves the constrained efficiency. This can be easily verified by setting $\rho = \gamma$ in eq.(8) and (9). The idea is that a higher inflation lowers the rate of return on real money balances used both as a savings instrument and a MOE for purchasing the special good. Thus, money creation generates dynamic inefficiency

in a sense that households save less and under-consume special goods compared to the social optimum.

Next, we consider the second case $\rho > \gamma$, which brings about a much richer set of comparative static analyses on stationary allocations. The next proposition summarizes such results.

Proposition 3 *Consider the second case where $\rho > \gamma$. Let Z_{EZ}^c and Z_{EZ}^s denote real money balances held for savings and purchasing special goods respectively in stationary equilibrium. Likewise, let $c_{y,EZ}$ denote general goods consumed by young workers in stationary equilibrium. A unique stationary monetary equilibrium exists, i.e., $\exists! Z_{EZ}^c$ and $\exists! Z_{EZ}^s$, only if $\ln \mu < \bar{\pi}$,*

$$\bar{\pi} = \frac{\gamma - \rho}{1 - \gamma} \left\{ \ln \left[\frac{(y + b)^{2-\gamma} - (y - b)^{2-\gamma}}{2b(2 - \gamma)} \right] \right\} - \gamma \ln y + \rho \ln (x - f(0)), \text{ and}$$

$$f(0) = \exp \left\{ \frac{\gamma}{\rho} \ln (y) + \frac{\rho - \gamma}{\rho(1 - \gamma)} \left\{ \ln \left[\frac{(y + b)^{2-\gamma} - (y - b)^{2-\gamma}}{2b(2 - \gamma)} \right] \right\} \right\}.$$

The followings then hold true in the unique stationary monetary equilibrium: $\partial Z_{EZ}^c / \partial b > 0$, $\partial Z_{EZ}^s / \partial b < 0$, $\partial Z_{EZ}^s / \mu < 0$, $\partial Z_{EZ}^c / \mu < 0$, $\partial c_{y,EZ} / \partial b < 0$, and $\partial c_{y,EZ} / \partial \mu > 0$. Lastly, the Friedman rule does not achieve the constrained efficiency due to uncertain endowments to old workers. Furthermore, even the second-best is not generally guaranteed by the Friedman rule. That is the optimal inflation rate that achieves the second-best critically depends upon structural parameters of the economy such as x, y , and b .

Proof. See the appendix. ■

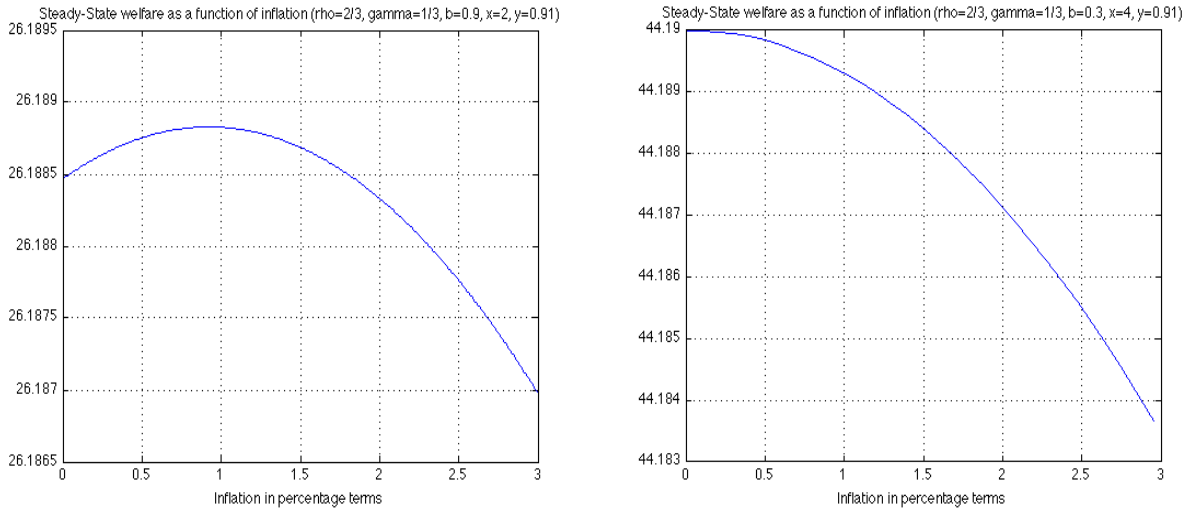
Focusing on the unique stationary monetary equilibrium, uncertainty effects are similar to the previous model. Since households dislike intertemporal inequality in terms of general good consumption to a greater extent, they accumulate more money savings in response to a higher b , i.e., $\partial Z_{EZ}^c / \partial b > 0$. This in turn means that young workers under-consume general goods in equilibrium. Given that households equalize the marginal utility from consuming general goods when young and special goods when old, cash holdings for special goods must fall as well when b goes up, i.e., $\partial Z_{EZ}^s / \partial b < 0$.

As before, inflation leads to lower real money balances used for both savings and purchasing special goods due to a lower rate of return on money holdings, i.e., $\partial Z_{EZ}^s / \mu < 0$ and $\partial Z_{EZ}^c / \mu < 0$. What is distinct from the previous model is that money creation and uncertainty create dynamic inefficiency in opposite direction only in terms of general goods consumed in equilibrium, i.e., $\partial Z_{EZ}^c / \partial b > 0$ and $\partial Z_{EZ}^c / \mu < 0$. On the contrary, inflation and uncertainty change the equilibrium special good consumption in the same direction, $\partial Z_{EZ}^s / \partial b < 0$ and $\partial Z_{EZ}^c / \mu < 0$. This implies that money creation can never correct the

dynamic inefficiency caused by a lower Z_{EZ}^s with a positive b . This intuitively explains why the *Friedman rule* can never achieve the constrained efficiency whenever uncertainty prevails in this economy.

What's more interesting is that the optimal (second-best) inflation rate in this case is usually not the *Friedman rule*. This follows from two offsetting welfare effects of inflation. This economy faces a fundamental trade-off between a negative price effect and a positive redistributive effect of inflation. The former refers to the ability of inflation affecting self-insurance through changes in the rate of return on currency, while the latter revolves around the ability of inflation providing risk sharing among agents through intergenerational transfers of money. For instance, the price effect of inflation on welfare is always negative since a higher inflation always leads to lower rates of return on both money savings and cash holdings for purchasing special goods. Therefore, other things being equal, the *Friedman rule* is the best policy to pursue. However, inflation also has a redistributive effect. This welfare effect could be positive as long as the level of μ is under a certain threshold level. This can be intuitively understood by the fact that $c_{y,EZ} < c_y$ and $c_{o,EZ} > c_o$ in equilibrium, and a higher inflation can move both $c_{y,EZ}$ and $c_{o,EZ}$ closer to the constrained efficient allocation up to a certain level, i.e., $\mu \uparrow \Rightarrow c_{y,EZ} \uparrow$ and $c_{o,EZ} \downarrow$. In sum, when the distribution effect dominates the price effect, the social welfare could increase up to a certain inflation rate, μ^{op} . Otherwise, zero inflation guarantees the second best.

Figure 3: Numerical examples for welfare analysis in the extended model with $\rho > \gamma$



Conditions under which the distribution effect dominates generally depend upon parameters of the economy. Figure 3 illustrates a couple of numerical examples. The left panel shows the case where the ratio between x and y is relatively small and the level of uncertainty

b is relatively high, while the right panel shows the opposite case. As seen, the former case implies a positive optimal inflation rate, on the other hand, the *Friedman rule* is optimal in the latter. Intuition behind this goes as follows. The distribution effect generally dominates the price effect whenever $c_{o,EZ} - c_{y,EZ}$ is larger because the marginal effect of a higher inflation on reducing $c_{o,EZ} - c_{y,EZ}$ gets bigger. Loosely speaking, when the intergenerational distribution of $\{c_{y,EZ}, c_{o,EZ}\}$ was so distorted to begin with, the positive redistributive effect of inflation becomes much more powerful. Thus, the distribution effect can dominate the price effect when inflation is relatively low. Since a lower (x/y) and a higher b both mean a more distorted intergenerational distribution of the general good consumption, a higher inflation can enhance social welfare up to a threshold point. Furthermore, it turns out that this threshold point, i.e., μ^{op} is increasing in the level of uncertainty once the μ^{op} is positive. This is again because a higher b leads to a more distorted distribution of $\{c_{y,EZ}, c_{o,EZ}\}$, strengthening the positive redistributive effect of inflation.

Using the intuition so far, it follows easily that the third case, $\rho < \gamma$, brings about opposite comparative static analyses on stationary allocations in general. First, Proposition 4 summarizes such results.

Proposition 4 *Consider the third case where $\rho < \gamma$. A unique stationary monetary equilibrium exists, i.e., $\exists! Z_{EZ}^c$ and $\exists! Z_{EZ}^s$, only if $\ln \mu < \bar{\pi}$. The followings then hold true in the unique stationary monetary equilibrium: $\partial Z_{EZ}^c / \partial b < 0$, $\partial Z_{EZ}^s / \partial b > 0$, $\partial Z_{EZ}^s / \mu < 0$, $\partial Z_{EZ}^c / \mu < 0$, $\partial c_{y,EZ} / \partial b > 0$, and $\partial c_{y,EZ} / \partial \mu > 0$. Lastly, the Friedman rule does not achieve the constrained efficiency due to uncertainty-induced dynamic inefficiency. Furthermore, even the second-best is not generally guaranteed by the Friedman rule. That is the optimal inflation rate that achieves the second-best critically depends upon structural parameters of the economy such as x, y , and b .*

Proof. See the appendix. ■

Uncertainty effects on the stationary allocation are exactly opposite to the second case. Since households don't mind intertemporal inequality in terms of general good consumption much, but are very averse to cross-section variation in the general good consumption when old, they accumulate *less* money savings in response to a higher b , i.e., $\partial Z_{EZ}^c / \partial b < 0$. This in turn means that young workers over-consume general goods in equilibrium. Given that households equalize the marginal utility from consuming general goods when young and special goods when old, cash holdings for special goods must increase as well when b goes up, i.e., $\partial Z_{EZ}^s / \partial b > 0$.

Inflation effects on the stationary allocation are same as before, i.e., $\partial Z_{EZ}^s / \mu < 0$ and $\partial Z_{EZ}^c / \mu < 0$, for obvious reasons. Interestingly, money creation here cannot restore the

constrained efficiency due to a different reason than the one in the second case economy. That is a higher b in this case causes dynamic inefficiency through lowering Z_{EZ}^c below the socially efficient level. However, a higher μ also lowers Z_{EZ}^c . In sum, the third case economy suffers from under-savings, but a higher inflation can only make things worse at least in this respect, i.e., a negative redistributive effect of inflation prevails. Thus, the constrained efficiency can never be achieved here.

What is crucial is that a positive level of inflation can achieve the second best under certain parameter values of the economy. This is in sharp contrast to the baseline model where the *Friedman rule* always leads to the second-best welfare outcome whenever $\rho < \gamma$. The outcome stems from the addition of special goods in the extended model. Recall that in the baseline model, the welfare effect of inflation was always negative because money creation and uncertainty generate dynamic inefficiency in the same direction. However, this is not the case here because at least a higher inflation can correct over-consumptions of the special good to some extent, i.e., $\partial Z_{EZ}^s / \partial b > 0$ and $\partial Z_{EZ}^s / \mu < 0$. In sum, there's a room for the *price* effect of inflation on welfare could be positive. This in turn means that the social welfare could increase up to a certain level of inflation rate when the positive price effect dominates the aforementioned negative redistributive effect.

Figure 4: Numerical examples for welfare analysis in the extended model with $\rho < \gamma$

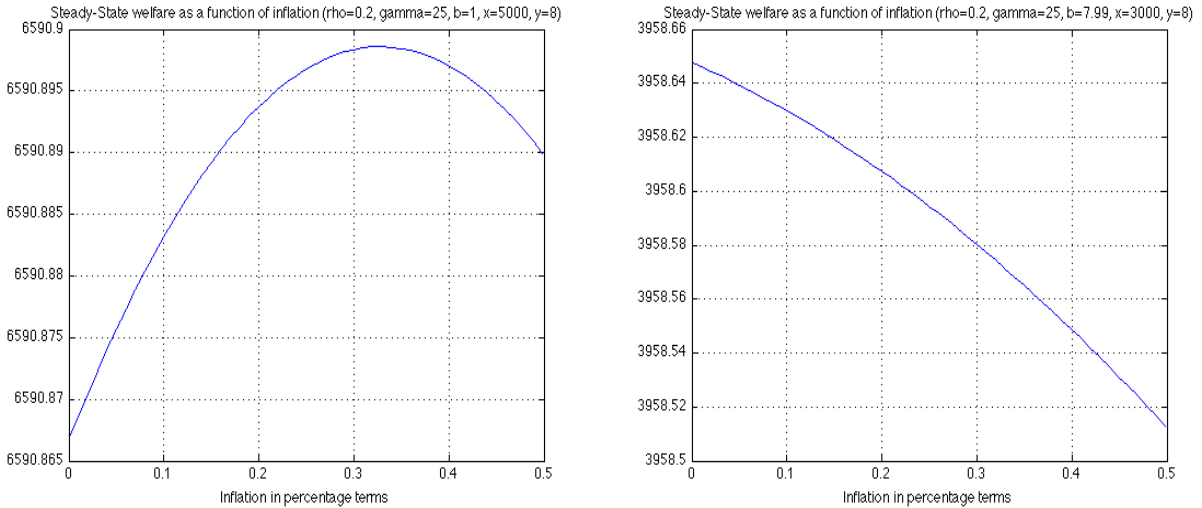


Figure 4 illustrate such examples in a numerical exercise. As opposed to Figure 3, the left panel shows the case where the ratio between x and y is relatively *high* and the level of uncertainty b is relatively *low*, while the right panel shows the opposite case. As can be witnessed, the former case implies a positive optimal inflation rate, on the other hand, the *Friedman rule* is optimal in the latter. Intuition behind this is exactly opposite to that in the

second case. The positive price effect generally dominates the negative redistributive effect when $c_{y,EZ} - c_{o,EZ}$ is larger and/or b is lower because the marginal (negative) redistributive effect of a higher inflation on raising $c_{y,EZ} - c_{o,EZ}$ gets smaller. Loosely speaking, when the intergenerational distribution of $\{c_{y,EZ}, c_{o,EZ}\}$ was relatively less distorted in the first place through a higher ratio between x and y and/or too much spending on special goods prevails in the first place, the positive price effect of inflation dominates. Thus, a higher inflation can enhance social welfare up to a threshold point. Lastly, a lower b implies a less distorted intergenerational distribution of the general good consumption, weakening the negative redistributive effect of inflation. Thus, the optimal inflation rate ought to *decrease* in aggregate output uncertainty once the former is positive. To sum up, the relationship between the optimal inflation rate and aggregate output uncertainty generally turns out to become positive (negative) when agents prefer late (early) resolution of uncertainty.

5 Conclusion

By exploiting a pure currency OLG model of money with the EZ preferences, this paper delivers a new set of monetary policy implications, especially in terms of a link between aggregate output uncertainty and the optimal inflation rate. I show that agents' preferences for the timing of uncertainty resolution and the initial condition for intertemporal misallocations crucially determine such relationship. I do admit limitations of this model for policy recommendations because economies considered here are extreme, e.g., too low transactional frequency and the infeasibility of any sort of taxation. Yet, I do believe that lessons from this study remain valid as a good benchmark for further studies, and it should be relatively easy to integrate real and more complex features of monetary economies into the model.

For example, following [Kocherlakota \(2005\)](#) and [Wallace \(2014\)](#), one could extend this model to include alternative instruments for intergenerational transfers, and study if such modifications alter the nature of the optimal monetary policy under aggregate uncertainty. Also, one could add other forms of institutions that mitigate various trading frictions such as financial assets and over-the-counter markets. This could allow us to study how liquidity properties of such additional institutions as well as the optimal monetary policy interact with aggregate uncertainty, and potentially to help us understand many asset pricing anomalies.⁷ Lastly, although not emphasized here, one could study dynamics and/or multiplicity in this framework. This dimension of research could potentially enrich types of multiple equilibria that the OLG model usually exhibits, e.g., [Boldrin and Woodford \(1990\)](#).

⁷ [Lagos \(2010\)](#), [Geromichalos, Herrenbrueck, and Salyer \(2013\)](#), and [Geromichalos, Lee, Lee, and Oikawa \(2015\)](#) are in similar spirits in this sense.

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Appendix

Proof for Lemma 2

Since $U(c_t, c_{t+1})$ is homogeneous of degree 1, Euler's theorem holds. Then using eq.(2),

$$U = \frac{\partial U}{\partial c_t} + E_t \left[\frac{\partial U}{\partial c_{t+1}} c_{t+1} \right]$$

where

$$\frac{\partial U}{\partial c_t} = \frac{1}{1-\rho} U^\rho (1-\rho) c_t^{-\rho} = U^\rho c_t^{-\rho} \quad (10)$$

$$\begin{aligned} \frac{\partial U}{\partial c_{t+1}} &= \frac{\partial U}{\partial R_t(c_{t+1})} \frac{\partial R_t(c_{t+1})}{\partial c_{t+1}} \\ &= U^\rho [R_t(c_{t+1})]^{-\rho} [R_t(c_{t+1})]^\gamma c_{t+1}^{-\gamma} \\ &= U^\rho [R_t(c_{t+1})]^{\gamma-\rho} c_{t+1}^{-\gamma}. \end{aligned} \quad (11)$$

Using eq.(10) and (11), IMRS or $Q_{t,t+1}$, which equals to U_2/U_1 , has the following closed form solution.

$$\begin{aligned} Q_{t,t+1} &\equiv \frac{U^\rho [R_t(c_{t+1})]^{\gamma-\rho} c_{t+1}^{-\gamma}}{U^\rho c_t^{-\rho}} \\ &= \left[\frac{c_{t+1}}{c_t} \right]^{-\rho} \left[\frac{c_{t+1}}{R_t(c_{t+1})} \right]^{\rho-\gamma}. \end{aligned}$$

Replacing c_{t+1} and c_t with $\varphi_{t+1}m_t + \tau_{t+1} + \varepsilon_{t+1}$ and $\varphi_t m_t$ respectively brings about $Q_{t,t+1}(m_t, \varepsilon_{t+1})$ in Lemma 2. Lastly, $E_t[(\varphi_{t+1}m_t + \tau_{t+1} + \varepsilon_{t+1})^{1-\gamma}]$ can be easily computed using $\mathcal{U}(y-b, y+b)$.

$$\begin{aligned} E_t[(h(m_t) + \varepsilon_{t+1})^{1-\gamma}] &= \int_{y-b}^{y+b} (h(m_t) + \varepsilon_{t+1})^{1-\gamma} \frac{1}{2b} d\varepsilon_{t+1} \\ &= \frac{1}{2b(2-\gamma)} (h(m_t) + y + b)^{(2-\gamma)} - (h(m_t) + y - b)^{(2-\gamma)}. \end{aligned} \quad (12)$$

Q.E.D.

Proof for Proposition 1

We first prove the existence and uniqueness of Z_{EZ} . The FOC in equilibrium becomes

as follows.

$$\ln \mu - \rho \ln(x - Z) + \gamma \ln(Z + y) = \frac{\gamma - \rho}{1 - \gamma} \left\{ \ln \left[\frac{(Z + y + b)^{2-\gamma} - (Z + y - b)^{2-\gamma}}{2b(2 - \gamma)} \right] \right\}$$

$$LHS(Z) = RHS(Z).$$

It's easy to see that $LHS' > 0$, and $\lim_{Z \rightarrow x} = \infty$. Now, RHS' has the following form.

$$RHS' = (\gamma - \rho) \underbrace{\left\{ \frac{(Z + y + b)^{1-\gamma} - (Z + y - b)^{1-\gamma}}{1 - \gamma} \right\}}_{>0} \underbrace{\left\{ \frac{(Z + y + b)^{2-\gamma} - (Z + y - b)^{2-\gamma}}{2 - \gamma} \right\}}_{>0}.$$

This implies that $RHS' < 0$ ($RHS' > 0$) if $\rho > \gamma$ ($\rho < \gamma$). Since the case 2 is based on $\rho > \gamma$, $RHS' < 0$. Thus, as long as $LHS(0) \leq RHS(0)$, $\exists! Z_{EZ}$. This proves the existence and uniqueness under $\ln \mu < \bar{\mu}$.

Next, we prove why $Z_{EZ} > Z_{SP}$. For this it suffices to show, first, $Z_{EZ}|_{b=0, \mu=1}$ is equal to or greater than Z_{SP} , and second, $\partial Z_{EZ} / \partial b > 0$. The first condition is easy to show from eq.(4). That is $Z + y = x - Z$, which implies $Z_{EZ} = (x - y)/2 \equiv Z_{SP}$. Second condition requires us to use Implicit Function Theorem (IFT) as follows.

$$\frac{\partial Z_{EZ}}{\partial b} = - \frac{\partial G / \partial b}{\partial G / \partial Z_{EZ}},$$

where

$$\frac{\partial G}{\partial Z_{EZ}} = \frac{\rho}{x - Z_{EZ}} + \frac{\gamma}{Z_{EZ} + y} - RHS' > 0 \quad \text{if } \rho > \gamma.$$

and

$$\begin{aligned} \frac{\partial G}{\partial b} &= - \frac{\gamma - \rho}{1 - \gamma} \left\{ (2 - \gamma) \frac{(Z_{EZ} + y + b)^{1-\gamma} - (Z_{EZ} + y - b)^{1-\gamma}}{(Z_{EZ} + y + b)^{2-\gamma} - (Z_{EZ} + y - b)^{2-\gamma}} - \frac{1}{b} \right\} \\ &\equiv - \frac{\gamma - \rho}{1 - \gamma} H. \end{aligned}$$

Now, we prove that H is always positive (negative) given that $\gamma > 1$ ($\gamma < 1$). First, given $\gamma > 1$ and $\mathcal{U}(y - b, y + b)$,

$$E_t [(Z_{EZ} + \varepsilon_{t+1})^{1-\gamma}] < \frac{(Z_{EZ} + y - b)^{1-\gamma} + (Z_{EZ} + y + b)^{1-\gamma}}{2}.$$

From eq.(12), this inequality can be re-expressed as below.

$$\frac{(Z_{EZ} + y - b)^{2-\gamma} + (Z_{EZ} + y - b)^{2-\gamma}}{2b(2-\gamma)} < \frac{(Z_{EZ} + y - b)^{1-\gamma} + (Z_{EZ} + y - b)^{1-\gamma}}{2},$$

$$\frac{1}{b} < (2-\gamma) \frac{(Z_{EZ} + y + b)^{1-\gamma} - (Z_{EZ} + y - b)^{1-\gamma}}{(Z_{EZ} + y + b)^{2-\gamma} - (Z_{EZ} + y - b)^{2-\gamma}}.$$

When $\gamma < 1$, all inequality signs so far get reversed. This proves why $H > 0$ ($H < 0$) when $\gamma > 1$ ($\gamma < 1$). Consequently,

$$\frac{\partial G}{\partial b} \begin{cases} > 0 & \text{if } \rho < \gamma \\ < 0 & \text{if } \rho > \gamma. \end{cases} \quad (13)$$

Combine this with $\partial G / \partial Z_{EZ} > 0$ under $\rho > \gamma$. Then, finally $\partial Z_{EZ} / \partial b > 0$, and $Z_{EZ} > Z_{SP}$. $\partial Z_{EZ} / \partial \mu < 0$ can be easily proved using the IFT as below.

$$\frac{\partial Z_{EZ}}{\partial \mu} = -\frac{\partial G / \partial \mu}{\partial G / \partial Z_{EZ}} = -\frac{1/\mu}{\frac{\rho}{x-Z_{EZ}} + \frac{\gamma}{Z_{EZ}+y} - RHS'} < 0, \quad \text{if } \rho > \gamma.$$

Lastly, μ^* is μ such that $G((x-y)/2, \mu, x, y, b) = 0$. The fact that $\mu^* > 1$ comes from $Z_{EZ}|_{\mu=1} > Z_{SP}$ and $\partial Z_{EZ} / \partial \mu < 0$. Given the closed form solution to μ^* as well as $\rho > \gamma$, $\partial \mu^* / \partial b > 0$ and $\partial \mu^* / \partial y < 0$ are straightforward. Q.E.D.

Proof for Proposition 2

When $\rho < \gamma$, $RHS' > 0$ and $RHS(x)$ is a finite real number. Thus, as long as $LHS(0) \leq RHS(0)$, $\exists! Z_{EZ}$. This proves the existence and uniqueness under $\ln \mu < \bar{\mu}$. If instead $LHS(0) \geq RHS(0)$ then, depending on the slope of RHS there could be multiple Z_{EZ} or non Z_{EZ} , and therefore, multiple monetary equilibrium or non-monetary equilibrium.

Next, we prove why $Z_{EZ} < Z_{SP}$ under the unique stationary monetary equilibrium case. For this it suffices to show, first, $Z_{EZ}|_{b=0, \mu=1}$ is equal to or less than Z_{SP} , and second, $\partial Z_{EZ} / \partial b < 0$. The first condition is easy to show from eq.(4). That is $Z + y = x - Z$, which implies $Z_{EZ} = (x - y)/2 \equiv Z_{SP}$. Second condition requires us to use Implicit Function Theorem (IFT) as follows.

$$\frac{\partial Z_{EZ}}{\partial b} = -\frac{\partial G / \partial b}{\partial G / \partial Z_{EZ}},$$

Since we know $\partial G / \partial b > 0$ from eq.(13) we only need to know the sign of $\partial G / \partial Z_{EZ}$. But we know for sure that $\partial Z_{EZ} / \partial \mu < 0$ since a higher μ would only shift the LHS upward. This

implies $\partial G/\partial Z_{EZ} > 0$ because

$$\frac{\partial Z_{EZ}}{\partial \mu} = -\frac{\partial G/\partial \mu}{\partial G/\partial Z_{EZ}} = -\frac{1/\mu}{\partial G/\partial Z_{EZ}} < 0.$$

Thus, $\partial Z_{EZ}/\partial b < 0$, and therefore $Z_{EZ} < Z_{SP}$. Finally, the fact that the *Friedman rule* does not achieve the Z_{SP} under $b > 0$ can be easily seen through $Z_{EZ}|_{b=0, \mu=1} = Z_{SP}$ and $\partial Z_{EZ}/\partial b < 0$. Further, due to $\partial Z_{EZ}/\partial \mu < 0$ the *Friedman rule* achieves the second-best. Q.E.D.

Proof for Proposition 3

We begin with proving the existence and uniqueness of $\{Z_{EZ}^c, Z_{EZ}^s\}$. We define a function $f(Z_{EZ}^c)$ such that $f(Z_{EZ}^c)$ equals to Z_{EZ}^s solving for eq.(9) given Z_{EZ}^c . Given that the LHS of e.q.(9) is increasing in Z_{EZ}^s and the RHS of eq.(9) is invariant to Z_{EZ}^s , it is easy to see that this function $f(Z_{EZ}^c)$ exists only if $\rho \ln(x - Z_{EZ}^c) > \gamma \ln(Z_{EZ}^c + y) + (\rho - \gamma) \ln R(Z_{EZ}^c + y)$, which in turn holds true as long as eq.(8) holds true. This also implies that $f(Z_{EZ}^c) < x, \forall Z_{EZ}^c \leq x$. Next, we prove $f'(Z_{EZ}^c) > 0$. For this, it suffices to show that the RHS of eq.(9) increases in Z_{EZ}^c . Since $\rho > \gamma$ and the certainty equivalent of future consumption value, i.e., $R(y + Z_{EZ}^c)$, is decreasing in Z_{EZ}^c , the RHS of eq.(9) is indeed increasing in Z_{EZ}^c . Lastly, by replacing Z_{EZ}^c with 0 one can get

$$f(0) = \exp \left\{ \frac{\gamma}{\rho} \ln(y) + \frac{\rho - \gamma}{\rho} \ln[R(y)] \right\} > 0.$$

By replacing $R(y)$ with structural parameters as in eq.(12), one can finally get $f(0)$ in Proposition 3.

Similar to $f(Z_{EZ}^c)$, we also define $g(Z_{EZ}^s)$ such that $g(Z_{EZ}^s)$ equals to Z_{EZ}^c solving for eq.(8) given Z_{EZ}^s . Note that the LHS of eq.(8) is increasing in Z_{EZ}^c with $LHS|_{Z_{EZ}^c=x-Z_{EZ}^s} \rightarrow \infty$ and the RHS of eq.(8) is decreasing in Z_{EZ}^c . Therefore, the function $g(Z_{EZ}^s)$ exists only if

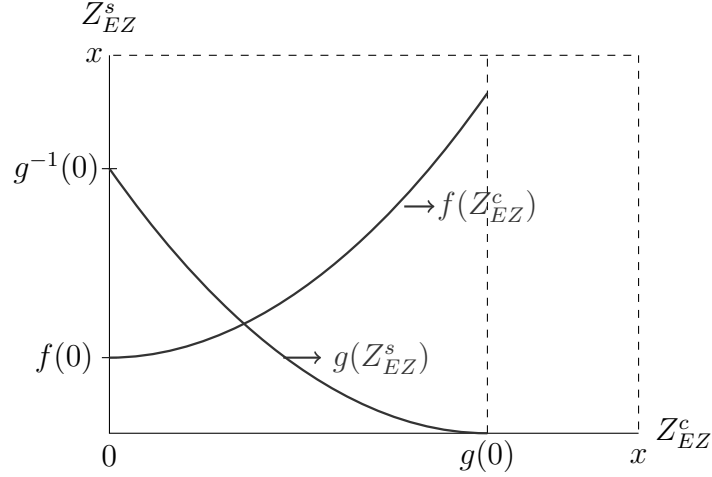
$$\ln \mu < (\gamma - \rho) \ln(R(y)) - \gamma \ln(y) + \rho \ln(x - Z_{EZ}^s|_{Z_{EZ}^c=0}). \quad (14)$$

By replacing $R(y)$ and $Z_{EZ}^s|_{Z_{EZ}^c=0}$ with the one in the $f(Z_{EZ}^c)$ and $f(0)$ respectively, one could finally get the $\bar{\pi}$ in Proposition 3. Next, we prove $g'(Z_{EZ}^s) < 0$. For this, it suffices to show that the LHS of eq.(8) increases in Z_{EZ}^s , which is trivial. Moreover, $g(0)$ is a finite real number and less than x due to eq.(14) and $LHS|_{Z_{EZ}^c=x-Z_{EZ}^s} \rightarrow \infty$ from eq.(8). Also, $g^{-1}(0) > f(0)$ due to eq.(14) and the fact that $g^{-1}(0)$ must satisfy the following.

$$\ln \mu = (\gamma - \rho) \ln(R(y)) - \gamma \ln(y) + \rho \ln(x - g^{-1}(0)).$$

The fact that $g^{-1}(0) < x$ is trivial. In sum, this proves that $\exists! Z_{EZ}^c$ and $\exists! Z_{EZ}^s$, only if $\ln \mu < \bar{\pi}$. Graphically, the unique equilibrium can be illustrated as below. Figure 5 greatly

Figure 5: A unique stationary equilibrium



helps us understand various comparative static analyses intuitively. First, $g(Z_{EZ}^s)$ shifts inward in response to an increase in μ . This is easily understood since the LHS of eq.(8) increases in μ , which in turn causes Z_{EZ}^c to fall given every level of Z_{EZ}^s . Plus, changes in μ have no effect on the $f(Z_{EZ}^c)$. Altogether, this confirms $\partial Z_{EZ}^s / \mu < 0$ and $\partial Z_{EZ}^c / \mu < 0$.

$g(Z_{EZ}^s)$ shifts outward in response to an increase in b . Again, this can be understood from eq.(8). Only the RHS of the equation increases in b due to $\rho > \gamma$ and the fact that $R(y)$ falls as b increases. $f(Z_{EZ}^c)$ also shifts outward as b increases. From the RHS of the eq.(9) with $\rho > \gamma$ and $\partial R(y) / \partial b < 0$ one can easily see that Z_{EZ}^s should fall given every level of Z_{EZ}^c when b increases. This proves $\partial Z_{EZ}^c / \partial b > 0$.

As for the $\partial Z_{EZ}^s / \partial b$, the graphical interpretation is limited since both curves shifts outward. Thus, we exploit the Cramer's rule. First, we define $V(b, Z_{EZ}^c, Z_{EZ}^s)$ as LHS of eq.(8) minus RHS of eq.(8), while $W(b, Z_{EZ}^c, Z_{EZ}^s)$ as LHS of eq.(9) minus RHS of eq.(9).

$$\frac{\partial Z_{EZ}^s}{\partial b} = \frac{\begin{vmatrix} -\frac{\partial V}{\partial b} & \frac{\partial V}{\partial Z_{EZ}^c} \\ -\frac{\partial W}{\partial b} & \frac{\partial W}{\partial Z_{EZ}^c} \end{vmatrix}}{\begin{vmatrix} -\frac{\partial V}{\partial Z_{EZ}^s} & \frac{\partial V}{\partial Z_{EZ}^c} \\ -\frac{\partial W}{\partial Z_{EZ}^s} & \frac{\partial W}{\partial Z_{EZ}^c} \end{vmatrix}} = \frac{\begin{vmatrix} \oplus & \oplus \\ \ominus & \ominus \end{vmatrix}}{\begin{vmatrix} \oplus & \oplus \\ \oplus & \ominus \end{vmatrix}} = \frac{-\frac{\partial V}{\partial b} \frac{\partial W}{\partial Z_{EZ}^c} + \frac{\partial V}{\partial Z_{EZ}^c} \frac{\partial W}{\partial b}}{\ominus} < 0, \quad \text{if } \rho > \gamma. \quad (15)$$

Equation (15) follows from below.

$$\begin{aligned}
\frac{\partial V}{\partial b} &= (\rho - \gamma) \frac{\partial R(b)/\partial b}{R(b)} \begin{cases} > 0 & \text{if } \rho < \gamma \\ < 0 & \text{if } \rho > \gamma. \end{cases} \\
\frac{\partial W}{\partial b} &= -\frac{(\rho - \gamma)}{\rho} \frac{\partial R(b)/\partial b}{R(b)} \begin{cases} < 0 & \text{if } \rho < \gamma \\ > 0 & \text{if } \rho > \gamma. \end{cases} \\
\frac{\partial V}{\partial Z_{EZ}^c} &= \frac{\rho}{x - Z_{EZ}^c - Z_{EZ}^s} + \frac{\gamma}{Z_{EZ}^c + y} + (\rho - \gamma) \frac{\partial R(Z_{EZ}^c)/\partial Z_{EZ}^c}{R(Z_{EZ}^c)} \begin{cases} \text{ambiguous} & \text{if } \rho < \gamma \\ > 0 & \text{if } \rho > \gamma. \end{cases} \\
\frac{\partial W}{\partial Z_{EZ}^c} &= -\frac{\gamma}{\rho(Z_{EZ}^c + y)} - \frac{(\rho - \gamma)}{\rho} \frac{\partial R(Z_{EZ}^c)/\partial Z_{EZ}^c}{R(Z_{EZ}^c)} \begin{cases} \text{ambiguous} & \text{if } \rho < \gamma \\ < 0 & \text{if } \rho > \gamma. \end{cases} \\
\frac{\partial V}{\partial Z_{EZ}^s} &= \frac{\rho}{x - Z_{EZ}^c - Z_{EZ}^s}. \\
\frac{\partial W}{\partial Z_{EZ}^s} &= \frac{1}{Z_{EZ}^s}. \\
-\frac{\partial V}{\partial b} \frac{\partial W}{\partial Z_{EZ}^c} + \frac{\partial V}{\partial Z_{EZ}^c} \frac{\partial W}{\partial b} &= -(\rho - \gamma) \frac{\partial R(b)/\partial b}{(x - Z_{EZ}^c - Z_{EZ}^s)R(b)} \begin{cases} < 0 & \text{if } \rho < \gamma \\ > 0 & \text{if } \rho > \gamma. \end{cases}
\end{aligned}$$

As for the effect of inflation on general goods consumed by young workers, $\partial c_{y,EZ}/\partial \mu > 0$ follows easily from $\partial Z_{EZ}^s/\mu < 0$ and $\partial Z_{EZ}^c/\mu < 0$. $\partial c_{y,EZ}/\partial b < 0$ follows from two facts. First, a higher inflation reduces special goods consumed by old shoppers in stationary equilibrium, i.e., $\partial Z_{EZ}^s/\partial \mu < 0$. Second, the marginal utility from consuming general goods by young workers should equal to that from consuming special goods by old shoppers in stationary equilibrium, i.e., eq.(8) and (9).

Lastly, the fact that the *Friedman rule* couldn't achieve the constrained efficient allocation can be understood with Figure 5. The constrained efficient allocation can be thought of a point where g and f functions intercept each other when $\mu = 1$ and $b = 0$. As explained before, as soon as b exceeds 0, both g and f functions shift outwards. However, changes in μ can only shift the g curve in Figure 5. Thus, the initial equilibrium point can never be restored by changes in μ in any direction. To figure out the (second-best) optimal inflation rate in this case, we construct a social welfare function, \mathcal{W} , as the sum (with equal weights) of all agents' net utilities in a unique stationary monetary equilibrium.

$$\mathcal{W} = (c_{y,EZ}^{1-\rho} + s_{EZ}^{1-\rho} + c_{o,EZ}^{1-\rho})^{1/(1-\rho)}.$$

Then the first derivative of \mathcal{W} with respect to μ is given by

$$\frac{\partial \mathcal{W}}{\partial \mu} = (c_{y,EZ}^{1-\rho} + s_{EZ}^{1-\rho} + c_{o,EZ}^{1-\rho})^{1/(1-\rho)} \left[\underbrace{s_{EZ}^{-\rho} \frac{\partial s_{EZ}}{\partial \mu}}_{<0} + \underbrace{c_{y,EZ}^{-\rho} \frac{\partial c_{y,EZ}}{\partial \mu}}_{>0} + \underbrace{c_{o,EZ}^{-\rho} \frac{\partial c_{o,EZ}}{\partial \mu}}_{<0} \right],$$

and ambiguous. Q.E.D.

Proof for Proposition 4

Similar to Proposition 3, we begin with a proof for the existence and uniqueness of $\{Z_{EZ}^c, Z_{EZ}^s\}$. Now, we define $f(Z_{EZ}^c)$ as before. Following the proof for Proposition 3, $f(Z_{EZ}^c)$ exists, and $f(Z_{EZ}^c) < x, \forall Z_{EZ}^c \leq x$. For $f'(Z_{EZ}^c) > 0$, one would have to confirm that the RHS of eq.(9) is increasing in Z_{EZ}^c even if $\rho < \gamma$. This is easy to show since $\ln(Z_{EZ}^c) > \ln(R(Z_{EZ}^c + y))$. The properties of $g(Z_{EZ}^s)$ are same as in Proposition 3. In sum, this proves that $\exists! Z_{EZ}^c$ and $\exists! Z_{EZ}^s$, only if $\ln \mu < \bar{\pi}$. Graphically, the unique equilibrium can be still illustrated by Figure 5.

Now, we prove various comparative static analyses. First, the fact that $g(Z_{EZ}^s)$ shifts inward in response to an increase in μ and $\partial Z_{EZ}^s/\mu < 0$ and $\partial Z_{EZ}^c/\mu < 0$ follows the same proof as in Proposition 3.

What is different from the previous proposition is that $g(Z_{EZ}^s)$, this time, shifts inward in response to an increase in b . Again, this can be understood from eq.(8). Only the RHS of the equation decreases in b due to $\rho < \gamma$ and the fact that $R(y)$ falls as b increases. $f(Z_{EZ}^c)$ also shifts inward as b increases. From the RHS of the eq.(9) with $\rho < \gamma$ and $\partial R(y)/\partial b < 0$ one can easily see that Z_{EZ}^s should increase given every level of Z_{EZ}^c when b increases. This proves $\partial Z_{EZ}^c/\partial b < 0$.

As for the $\partial Z_{EZ}^s/\partial b$, the graphical interpretation is limited since both curves shifts inward. Thus, we exploit the Cramer's rule. As in the proof for Proposition 3, we define $V(b, Z_{EZ}^c, Z_{EZ}^s)$ as LHS of eq.(8) minus RHS of eq.(8), while $W(b, Z_{EZ}^c, Z_{EZ}^s)$ as LHS of eq.(9) minus RHS of eq.(9).

$$\frac{\partial Z_{EZ}^s}{\partial b} = \frac{\begin{vmatrix} -\frac{\partial V}{\partial b} & \frac{\partial V}{\partial Z_{EZ}^c} \\ -\frac{\partial W}{\partial b} & \frac{\partial W}{\partial Z_{EZ}^c} \end{vmatrix}}{\begin{vmatrix} -\frac{\partial V}{\partial Z_{EZ}^s} & \frac{\partial V}{\partial Z_{EZ}^c} \\ -\frac{\partial W}{\partial Z_{EZ}^s} & \frac{\partial W}{\partial Z_{EZ}^c} \end{vmatrix}} = \frac{-\frac{\partial V}{\partial b} \frac{\partial W}{\partial Z_{EZ}^c} + \frac{\partial V}{\partial Z_{EZ}^c} \frac{\partial W}{\partial b}}{\frac{\partial V}{\partial Z_{EZ}^s} \frac{\partial W}{\partial Z_{EZ}^c} - \frac{\partial W}{\partial Z_{EZ}^s} \frac{\partial V}{\partial Z_{EZ}^c}} > 0, \quad \text{if } \rho < \gamma. \quad (16)$$

In eq.(16), the numerator becomes negative if $\rho < \gamma$. This follows from the previous propo-

sition's proof. The denominator is given by

$$\begin{aligned}
\frac{\partial V}{\partial Z_{EZ}^s} \frac{\partial W}{\partial Z_{EZ}^c} - \frac{\partial W}{\partial Z_{EZ}^s} \frac{\partial V}{\partial Z_{EZ}^c} &= \frac{-\rho}{Z_{EZ}^s(x - Z_{EZ}^c - Z_{EZ}^s)} \\
&\quad - \frac{1}{x - Z_{EZ}^c - Z_{EZ}^s} \left[\frac{\gamma}{Z_{EZ}^c + y} - \frac{\partial R(Z_{EZ}^c + y)/\partial Z_{EZ}^c}{R(Z_{EZ}^c + y)}(\gamma - \rho) \right] \\
&\quad - \frac{1}{Z_{EZ}^s} \left[\frac{\gamma}{Z_{EZ}^c + y} - \frac{\partial R(Z_{EZ}^c + y)/\partial Z_{EZ}^c}{R(Z_{EZ}^c + y)}(\gamma - \rho) \right] \\
&< 0 \text{ if } R(Z_{EZ}^c + y) > (\partial R(Z_{EZ}^c + y)/\partial Z_{EZ}^c)(Z_{EZ}^c + y)
\end{aligned}$$

where the last condition holds due to the concavity of the R function. Lastly, proofs for statements regarding the optimal inflation rate are identical to those in the proof for Proposition 3. Q.E.D.